Leibniz の積分公式

定理 1 Leibniz の積分公式 $(-\infty, \infty)$ で定義された連続な関数 $f(x,y)$ 及び $\dfrac{\partial}{\partial x}f(x,y)$，定数 a，b を考えるとき

$$\frac{d}{dx}\int_a^b f(x,y)dy = \int_a^b \frac{\partial}{\partial x}f(x,y)dy \tag{1}$$

[**証明**] $F_y(x,y) = f(x,y)$ とすると，

$$\begin{aligned}\frac{d}{dx}\int_a^b f(x,y)dy &= \frac{d}{dx}\int_a^b F_y(x,y)dy \\ &= \frac{d}{dx}\left[F(x,y)\right]_{y=a}^{y=b} \\ &= \frac{d}{dx}\left\{F(x,b) - F(x,a)\right\} \\ &= F_x(x,b) - F_x(x,a)\end{aligned}$$

$$\begin{aligned}\int_a^b \frac{\partial}{\partial x}f(x,y)dy &= \int_a^b F_{yx}dy \\ &= \int_a^b F_{xy}dy \\ &= \left[F_x(x,y)\right]_{y=a}^{y=b} \\ &= F_x(x,b) - F_x(x,a)\end{aligned}$$

[証明終わり]

[別証] $F_y(x,y) = f(x,y)$ とすると，

$$\begin{aligned}\int_a^b \frac{\partial}{\partial x}f(x,y)dy &= \int_a^b \lim_{h\to 0} f(x+h,y) - f(x,y)dy \\ &= \lim_{h\to 0}\frac{1}{h}\int_a^b \left\{f(x+h,y) - f(x,y)\right\}dy \\ &= \lim_{h\to 0}\frac{1}{h}\left\{F(x+h,b) - F(x+h,b) - F(x+h,a) + F(x+h,a)\right\} \\ &= F_x(x,b) - F_x(x,a) = \text{左辺}\end{aligned}$$

定理 2 一般化された Leibniz の積分公式
x の関数 $\alpha(x), \beta(x)$ を積分区間とすると，

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(x,y)dy = \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x}f(x,y)dy + \beta'(x)f(x,\beta(x)) - \alpha'(x)f(x,\alpha(x)) \tag{2}$$

[**証明**]

$$
\begin{aligned}
\frac{d}{dx}\int_{\alpha}^{\beta} f(x,y)dy &= \frac{d}{dx}\int \alpha^{\beta} F_y(x,y)dy \\
&= \frac{d}{dx}\left[F(x,y)\right]_{y=\alpha(x)}^{y=\beta(x)} \\
&= \frac{d}{dx}\left\{F(x,\beta(x)) - F(x,\alpha(x))\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}F(x,\alpha(x)) &= \lim_{h\to 0}\frac{F(x+h,\alpha(x+h)) - F(x,\alpha(x))}{h} \\
&= \lim_{h\to 0}\frac{F(x+h,\alpha(x+h)) - F(x,\alpha(x+h)) + F(x,\alpha(x+h)) - F(x,\alpha(x))}{h} \\
&= \lim_{h\to 0}\frac{F(x+h,\alpha(x+h)) - F(x,\alpha(x+h))}{h} + \lim_{h\to 0}\frac{F(x,\alpha(x+h)) - F(x,\alpha(x))}{h} \\
&= F_x(x,\alpha(x)) + \lim_{h\to 0}\frac{F(x,\alpha(x)+\alpha'(x)h) - F(x,\alpha(x))}{\alpha'(x)h} \times \alpha'(x) \\
&= F_x(x,\alpha(x)) + F_y(x,\alpha(x))\alpha'(x) \\
&= F_x(x,\alpha(x)) + f(x,\alpha(x))\alpha'(x) \qquad (3)
\end{aligned}
$$

同様に,

$$
\frac{d}{dx}F(x,\beta(x)) = F_x(x,\beta(x)) + f(x,\beta(x))\beta'(x) \qquad (4)
$$

$$
\begin{aligned}
\int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} f(x,y)dy &= \int_{\alpha(x)}^{\beta(x)} \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h}dy \\
&= \lim_{h\to 0}\frac{1}{h}\int_{\alpha(x)}^{\beta(x)} \left\{f(x+h,y) - f(x,y)\right\}dy \\
&= \lim_{h\to 0}\frac{1}{h}\left\{F(x+h,\beta(x)) - F(x,\beta(x)) - F(x+h,\alpha(x)) + F(x,\alpha(x))\right\} \\
&= F_x(x,\beta(x)) - F_x(x,\alpha(x)) \qquad (5)
\end{aligned}
$$

(3),(4),(5) より題意は証明された.