コーシー・リーマン方程式

定理 1

$$z = x + iy, f(z) = u + iv$$

のとき，$f(z)$ が微分可能である条件は

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

である．

［証明］全微分の公式より

$$\begin{aligned}
\Delta u &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y \\
\Delta v &= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y \\
\therefore \Delta f &= \Delta u + i\Delta v \\
&= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + i\frac{\partial v}{\partial x}\Delta x + i\frac{\partial v}{\partial y}\Delta y \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y
\end{aligned}$$

$\Delta z = \Delta r(\cos\theta + i\sin\theta) = \Delta r e^{i\theta}$ とおくと

$$\begin{aligned}
\Delta f &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta r\cos\theta + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta r\sin\theta \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{\Delta z}{e^{i\theta}}\cos\theta + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{\Delta z}{e^{i\theta}}\sin\theta \\
\therefore \frac{df}{dz} &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)e^{-i\theta}\cos\theta + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)e^{-i\theta}\sin\theta \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)e^{-i\theta}(e^{i\theta} - i\sin\theta) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)e^{-i\theta}\sin\theta \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)e^{-i\theta}\sin\theta \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right\}e^{-i\theta}\sin\theta
\end{aligned}$$

$\dfrac{df}{dz}$ が Δz の偏角 θ によらないという条件は,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

［証明おわり］

参考文献

[1] 小野寺嘉孝『なっとくする複素関数』（講談社，2000 年）