正規分布の標本の分散の期待値 (2)

問題 1 $N(\mu, \sigma^2)$ から n 個の標本をとった場合の分散の期待値を求めよ.

[別解] 標本の分散を S^2 とすると,

$$\begin{aligned} nE\left[S^2\right] &= E\left[nS^2\right] \\ &= E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] \\ &= E\left[\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X}^2)\right] \\ &= E\left[\sum_{i=1}^{n}X_i^2 - 2\sum_{i=1}^{n}X_i\bar{X} + \sum_{i=1}^{n}\bar{X}^2\right] \\ &= E\left[\sum_{i=1}^{n}X_i^2 - 2\bar{X}\sum_{i=1}^{n}X_i + n\bar{X}^2\right] \end{aligned}$$

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ を代入して,

$$\begin{aligned} nE\left[S^2\right] &= E\left[\sum_{i=1}^{n}X_i^2 - 2\cdot\frac{1}{n}\sum_{i=1}^{n}X_i\sum_{i=1}^{n}X_i + n\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)^2\right] \\ &= E\left[\sum_{i=1}^{n}X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n}X_i\right)^2\right] \qquad (1) \end{aligned}$$

$$E\left[\sum_{i=1}^{n}X_i^2\right] = n(\mu^2 + \sigma^2) \qquad (2)$$

$$\begin{aligned} E\left[\left(\sum_{i=1}^{n}X_i\right)^2\right] &= E\left[\left(\sum_{i=1}^{n}X_i^2 + \sum_{i\neq J}X_iX_j\right)\right] \\ &= E\left[\sum_{i=1}^{n}X_i^2\right] + E\left[\sum_{i\neq J}X_iX_j\right] \\ &= n(\mu^2 + \sigma^2) + E\left[\sum_{i\neq J}X_iX_j\right] \\ &= n(\mu^2 + \sigma^2) + (n^2 - n)E\left[X_iX_j\right] \\ &= n(\mu^2 + \sigma^2) + (n^2 - n)E\left[X_i\right]E\left[X_j\right] \\ &= n(\mu^2 + \sigma^2) + (n^2 - n)\mu^2 \\ &= n\sigma^2 + n^2\mu^2 \qquad (3) \end{aligned}$$

(2),(3) を (1) に代入して,

$$\begin{aligned}
nE\left[S^2\right] &= E\left[\sum_{i=1}^{n} X_i^2 - 2\cdot\frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} X_i + n\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] \\
&= E\left[\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2\right] \\
&= n(\mu^2+\sigma^2) - (\sigma^2 + n\mu^2) \\
&= (n-1)\sigma^2 \\
E\left[S^2\right] &= \frac{(n-1)\sigma^2}{n} \cdots (Ans.)
\end{aligned}$$