正規分布の標本の分散の期待値

問題 1 $N(\mu, \sigma^2)$ から n 個の標本をとった場合の分散の期待値を求めよ.

[解]　標本の分散を S^2 とすると,

$$
\begin{aligned}
nE\left[S^2\right] &= E\left[nS^2\right] \\
&= E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] \\
&= E\left[\sum_{i=1}^{n}\left\{(X_i - \mu) + (\mu - \bar{X})\right\}^2\right] \\
&= E\left[\sum_{i=1}^{n}\left\{(X_i - \mu) - (\bar{X} - \mu)\right\}^2\right] \\
&= E\left[\sum_{i=1}^{n}\left\{(X_i - \mu)^2 - 2(X_i - \mu)(\bar{X} - \mu) + (\bar{X} - \mu)^2\right\}\right] \\
&= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 - 2\sum_{i=1}^{n}(X_i - \mu)(\bar{X} - \mu) + \sum_{i=1}^{n}(\bar{X} - \mu)^2\right] \\
&= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 - 2(\bar{X} - \mu)\sum_{i=1}^{n}(X_i - \mu) + \sum_{i=1}^{n}(\bar{X} - \mu)^2\right] \\
&= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 - 2n(\bar{X} - \mu)^2 + n(\bar{X} - \mu)^2\right] \\
&= E\left[\sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{X} - \mu)^2\right] \\
&= E\left[\sum_{i=1}^{n}(X_i - \mu)^2\right] - nE\left[(\bar{X} - \mu)^2\right] \\
&= n\sigma^2 - nE\left[(\bar{X} - \mu)^2\right] \qquad (1)
\end{aligned}
$$

ここで，少し戻ってしまうことになるが，

$$
\begin{aligned}
E\left[(\bar{X}-\mu)^2\right] &= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}X_i-\mu\right)^2\right] \\
&= \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n}X_i-n\mu\right)^2\right] \\
&= \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n}X_i\right)^2-2n\mu\sum_{i=1}^{n}X_i+n^2\mu^2\right] \\
&= \frac{1}{n^2}\left\{E\left[\left(\sum_{i=1}^{n}X_i\right)^2\right]-2n\mu E\left[\sum_{i=1}^{n}X_i\right]+n^2\mu^2\right\} \\
&= \frac{1}{n^2}\left\{E\left[\left(\sum_{i=1}^{n}X_i\right)^2\right]-n^2\mu^2\right\} \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
E\left[\left(\sum_{i=1}^{n}X_i\right)^2\right] &= E\left[\left(\sum_{i=1}^{n}X_i^2+\sum_{i\neq J}X_iX_j\right)\right] \\
&= E\left[\sum_{i=1}^{n}X_i^2\right]+E\left[\sum_{i\neq J}X_iX_j\right] \\
&= n(\mu^2+\sigma^2)+E\left[\sum_{i\neq J}X_iX_j\right] \\
&= n(\mu^2+\sigma^2)+(n^2-n)E\left[X_iX_j\right] \\
&= n(\mu^2+\sigma^2)+(n^2-n)E\left[X_i\right]E\left[X_j\right] \\
&= n(\mu^2+\sigma^2)+(n^2-n)\mu^2 \\
&= n\sigma^2+n^2\mu^2 \qquad (3)
\end{aligned}
$$

(3) を (2) に代入して，

$$
\begin{aligned}
E\left[(\bar{X}-\mu)^2\right] &= \frac{1}{n^2}(n\sigma^2+n^2\mu^2-n^2\mu^2) \\
&= \frac{1}{n}\sigma^2 \qquad (4)
\end{aligned}
$$

(4) を (1) に代入して，

$$
\begin{aligned}
nE\left[S^2\right] &= (n-1)\sigma^2 \\
E\left[S^2\right] &= \frac{(n-1)\sigma^2}{n}\cdots(Ans.)
\end{aligned}
$$

確率変数 X と Y が互いに独立であるときは，確率変数 XY の期待値は

E[XY]=E[X]E[Y]

$$
\begin{aligned}
E[XY] &= \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i)(x_j)P(X=x_i, Y=y_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i)(x_j)P(X=x_i)P(Y=y_j) \\
&= \sum_{i=1}^{n} x_i P(X=x_i)\left\{\sum_{j=1}^{m} x_j P(Y=y_j)\right\} \\
&= \left\{\sum_{i=1}^{n} x_i P(X=x_i)\right\}\left\{\sum_{j=1}^{m} x_j P(Y=y_j)\right\} \\
&= E[X]E[Y]
\end{aligned}
$$

参考文献

[1] 河村央也「青空学園数学科」<http://www33.ocn.ne.jp/~aozora_gakuen/>