標本平均の分散

問題 1 連続確率分布の n 個の標本の平均の分散は母分散の $\dfrac{1}{n}$ になることを証明せよ．

[解] 確率密度関数を $f(x)$，母平均を μ，母分散を σ^2 とすると，以下の式を証明すればよい．

$$\int_{\infty}^{\infty}\cdots\int_{\infty}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)^2\prod_{k=1}^{n}f(x_k)dx_1\cdots dx_n-\mu^2=\frac{\sigma^2}{n} \tag{1}$$

(I) n=1 で (1) が成り立つのは明らか．
(II) n で (1) が成り立つと仮定すると，

$$\begin{aligned}
&\int_{\infty}^{\infty}\cdots\int_{\infty}^{\infty}\left(\frac{1}{n+1}\sum_{k=1}^{n+1}x_k\right)^2\prod_{k=1}^{n+1}f(x_k)dx_1\cdots dx_{n+1}-\mu^2\\
=&\left(\frac{1}{n+1}\right)^2\int_{\infty}^{\infty}\cdots\int_{\infty}^{\infty}\left(\left(\sum_{k=1}^{n}x_k+x_{n+1}\right)\right)^2\prod_{k=1}^{n}f(x_k)f(x_{n+1})dx_1\cdots dx_{n+1}-\mu^2\\
=&\left(\frac{1}{n+1}\right)^2\int_{\infty}^{\infty}\cdots\int_{\infty}^{\infty}\left\{\left(\sum_{k=1}^{n}x_k\right)^2+2\left(\sum_{k=1}^{n}x_k\right)x_{n+1}+x_{n+1}^2\right\}\prod_{k=1}^{n}f(x_k)f(x_{n+1})dx_1\cdots dx_{n+1}-\mu^2
\end{aligned} \tag{2}$$

式が複雑になってきたので分割して考える．

$$\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{\infty}^{\infty}\left(\sum_{k=1}^{n}x_k\right)^2\prod_{k=1}^{n}f(x_k)f(x_{n+1})dx_1\cdots dx_{n+1}\\
=&\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\cdots\int_{\infty}^{\infty}\left(\sum_{k=1}^{n}x_k\right)^2\prod_{k=1}^{n}f(x_k)dx_1\cdots dx_n\right\}f(x_{n+1})dx_{n+1}\\
=&n^2\int_{-\infty}^{\infty}\left(\mu^2+\frac{\sigma^2}{n}\right)f(x_{n+1})dx_{n+1}\\
=&n^2\mu^2+n\sigma^2 \qquad (3)\\
&\int_{-\infty}^{\infty}\cdots\int_{\infty}^{\infty}\left(\sum_{k=1}^{n}x_k\right)x_{n+1}\prod_{k=1}^{n}f(x_k)f(x_{n+1})dx_1\cdots dx_{n+1}\\
=&\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\cdots\int_{\infty}^{\infty}\sum_{k=1}^{n}x_k\prod_{k=1}^{n}f(x_k)dx_1\cdots dx_n\right\}x_{n+1}f(x_{n+1})dx_{n+1}\\
&\int_{-\infty}^{\infty}\cdots\int_{\infty}^{\infty}\sum_{k=1}^{n}x_k\prod_{k=1}^{n}f(x_k)dx_1\cdots dx_n\\
=&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\sum_{k=1}^{n-1}x_k+\mu\right)\prod_{k=1}^{n}f(x_k)dx_1\cdots dx_{n-1}\\
\cdots&\\
=&n\mu
\end{aligned}$$

$$
\begin{aligned}
&\therefore \int_{-\infty}^{\infty} \cdots \int_{\infty}^{\infty} \left(\sum_{k=1}^{n} x_k \right) x_{n+1} \prod_{k=1}^{n} f(x_k) f(x_{n+1}) dx_1 \cdots dx_{n+1} \\
=& n\mu \int_{-\infty}^{\infty} x_{n+1} f(x_{n+1}) dx_{n+1} \\
=& n\mu^2 && (4)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \cdots \int_{\infty}^{\infty} x_{n+1}^2 \prod_{k=1}^{n} f(x_k) f(x_{n+1}) dx_1 \cdots dx_{n+1} \\
=& \int_{-\infty}^{\infty} x_{n+1}^2 \left\{ \int_{-\infty}^{\infty} \cdots \int_{\infty}^{\infty} \prod_{k=1}^{n} f(x_k) dx_1 \cdots dx_n \right\} f(x_{n+1}) dx_{n+1} \\
=& \int_{-\infty}^{\infty} x_{n+1}^2 f(x_{n+1}) dx_{n+1} \\
=& \mu^2 + \sigma^2 && (5)
\end{aligned}
$$

(3),(4),(5) を (2) に代入すると，

$$
\begin{aligned}
&\int_{\infty}^{\infty} \cdots \int_{\infty}^{\infty} \left(\frac{1}{n+1} \sum_{k=1}^{n+1} x_k \right)^2 \prod_{k=1}^{n+1} f(x_k) dx_1 \cdots dx_{n+1} - \mu^2 \\
=& \left(\frac{1}{n+1} \right)^2 \left(n^2\mu^2 + n\sigma^2 + 2n\mu^2 + \mu^2 + \sigma^2 \right) - \mu^2 \\
=& \left(\frac{1}{n+1} \right)^2 \left\{ \sigma^2(n+1) + \mu^2(n+1)^2 \right\} - \mu^2 \\
=& \frac{\sigma^2}{n+1}
\end{aligned}
$$

(I),(II) より，全ての自然数 n について (1) が成り立つ． [証明おわり]