

離散型確率分布の標本平均の分散

問題 1 N 個からなる離散型確率分布の n 個の標本の平均の分散を求めよ .

[解] 確率変数 x_k に対する確率を $f(x_k)$, 母平均を μ , 母分散を σ^2 とすると , 標本平均の分散 s^2 は ,

$$s^2 = \frac{1}{N C_n} \sum_{N C_n} \left(\frac{1}{n} \sum_{k=1}^n x_k \right)^2 \prod_{k=1}^n f(x_k) - \mu^2 \quad (1)$$

(I) $n=1$ で (1) が成り立つのは明らか .

(II) n で (1) が成り立つと仮定すると ,

$$\begin{aligned} & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{1}{n+1} \sum_{k=1}^{n+1} x_k \right)^2 \prod_{k=1}^{n+1} f(x_k) dx_1 \cdots dx_{n+1} - \mu^2 \\ &= \left(\frac{1}{n+1} \right)^2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sum_{k=1}^n x_k + x_{n+1} \right)^2 \prod_{k=1}^n f(x_k) f(x_{n+1}) dx_1 \cdots dx_{n+1} - \mu^2 \\ &= \left(\frac{1}{n+1} \right)^2 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \left(\sum_{k=1}^n x_k \right)^2 + 2 \left(\sum_{k=1}^n x_k \right) x_{n+1} + x_{n+1}^2 \right\} \prod_{k=1}^n f(x_k) f(x_{n+1}) dx_1 \cdots dx_{n+1} - \mu^2 \end{aligned} \quad (2)$$

式が複雑になってきたので分割して考える .

$$\begin{aligned} & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sum_{k=1}^n x_k \right)^2 \prod_{k=1}^n f(x_k) f(x_{n+1}) dx_1 \cdots dx_{n+1} \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sum_{k=1}^n x_k \right)^2 \prod_{k=1}^n f(x_k) dx_1 \cdots dx_n \right\} f(x_{n+1}) dx_{n+1} \\ &= n^2 \int_{-\infty}^{\infty} \left(\mu^2 + \frac{\sigma^2}{n} \right) f(x_{n+1}) dx_{n+1} \\ &= n^2 \mu^2 + n \sigma^2 \quad (3) \\ & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sum_{k=1}^n x_k \right) x_{n+1} \prod_{k=1}^n f(x_k) f(x_{n+1}) dx_1 \cdots dx_{n+1} \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{k=1}^n x_k \prod_{k=1}^n f(x_k) dx_1 \cdots dx_n \right\} x_{n+1} f(x_{n+1}) dx_{n+1} \end{aligned}$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{k=1}^n x_k \prod_{k=1}^n f(x_k) dx_1 \cdots dx_n \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sum_{k=1}^{n-1} x_k + \mu \right) \prod_{k=1}^n f(x_k) dx_1 \cdots dx_{n-1} \\
&\quad \dots \\
&= n\mu \\
& \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sum_{k=1}^n x_k \right) x_{n+1} \prod_{k=1}^n f(x_k) f(x_{n+1}) dx_1 \cdots dx_{n+1} \\
&= n\mu \int_{-\infty}^{\infty} x_{n+1} f(x_{n+1}) dx_{n+1} \\
&= n\mu^2 \tag{4}
\end{aligned}$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_{n+1}^2 \prod_{k=1}^n f(x_k) f(x_{n+1}) dx_1 \cdots dx_{n+1} \\
&= \int_{-\infty}^{\infty} x_{n+1}^2 \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^n f(x_k) dx_1 \cdots dx_n \right\} f(x_{n+1}) dx_{n+1} \\
&= \int_{-\infty}^{\infty} x_{n+1}^2 f(x_{n+1}) dx_{n+1} \\
&= \mu^2 + \sigma^2 \tag{5}
\end{aligned}$$

(4),(5),(6) を (3) に代入すると ,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{1}{n+1} \sum_{k=1}^{n+1} x_k \right)^2 \prod_{k=1}^{n+1} f(x_k) dx_1 \cdots dx_{n+1} - \mu^2 \\
&= \left(\frac{1}{n+1} \right)^2 (n^2 \mu^2 + n\sigma^2 + 2n\mu^2 + \mu^2 + \sigma^2) - \mu^2 \\
&= \left(\frac{1}{n+1} \right)^2 \{ \sigma^2(n+1) + \mu^2(n+1)^2 \} - \mu^2 \\
&= \frac{\sigma^2}{n+1}
\end{aligned}$$

(I),(II) より , 全ての自然数 n について (1) が成り立つ . [証明おわり]