数個の独立変数を含む関数に依存する汎関数

汎関数,

$$v(z(x,y)) = \iint_D F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial x}\right) dxdy$$

の極値を調べる．領域 D の境界を C とする．ここまでの汎関数と同じように，極値となるための必要条件は,

$$\delta v = \left[\frac{\partial}{\partial \alpha} v(z(x,y) + \alpha \delta z)\right]_{\alpha=0} = 0$$

である．

$$\begin{aligned}
\delta v &= \left[\frac{\partial}{\partial \alpha} \iint_D F\left(x, y, z + \alpha\delta z, \frac{\partial}{\partial x}(z + \alpha\delta z), \frac{\partial}{\partial y}(z + \alpha\delta z)\right) dxdy\right]_{\alpha=0} \\
&= \left[\iint_D \frac{\partial}{\partial \alpha} F\left(x, y, z + \alpha\delta z, \frac{\partial}{\partial x}(z + \alpha\delta z), \frac{\partial}{\partial y}(z + \alpha\delta z)\right) dxdy\right]_{\alpha=0} \\
&= \left[\iint_D \frac{\partial}{\partial \alpha} F\left(x, y, z + \alpha\delta z, \frac{\partial z}{\partial x} + \alpha\frac{\partial}{\partial x}\delta z, \frac{\partial z}{\partial y} + \alpha\frac{\partial}{\partial y}\delta z\right) dxdy\right]_{\alpha=0}
\end{aligned}$$

$Z = z + \alpha\delta z, P = \frac{\partial z}{\partial x} + \alpha\frac{\partial}{\partial x}\delta z, Q = \frac{\partial z}{\partial y} + \alpha\frac{\partial}{\partial y}\delta z$ とすると，

$$\begin{aligned}
\delta v &= \left[\iint_D \frac{\partial}{\partial \alpha} F(x, y, Z, P, Q) dxdy\right]_{\alpha=0} \\
&= \iint_D \left[\frac{\partial F}{\partial Z}\frac{\partial Z}{\partial \alpha} + \frac{\partial F}{\partial P}\frac{\partial P}{\partial \alpha} + \frac{\partial F}{\partial Q}\frac{\partial Q}{\partial \alpha}\right]_{\alpha=0} dxdy
\end{aligned}$$

$\frac{\partial Z}{\partial \alpha} = \delta z, \frac{\partial P}{\partial \alpha} = \frac{\partial}{\partial x}\delta z, \frac{\partial Q}{\partial \alpha} = \frac{\partial}{\partial y}\delta z$ なので，

$$\delta v = \iint_D \left[\frac{\partial F}{\partial Z}\delta z + \frac{\partial F}{\partial P}\frac{\partial}{\partial x}\delta z + \frac{\partial F}{\partial Q}\frac{\partial}{\partial y}\delta z\right]_{\alpha=0} dxdy$$

$\frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial P}\delta z\right\} = \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial P}\right\}\delta z + \frac{\partial F}{\partial P}\frac{\partial}{\partial x}\delta z, \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial Q}\delta z\right\} = \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial Q}\right\}\delta z + \frac{\partial F}{\partial Q}\frac{\partial}{\partial y}\delta z$ より，

$$\frac{\partial F}{\partial P}\frac{\partial}{\partial x}\delta z = \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial P}\delta z\right\} - \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial P}\right\}\delta z, \frac{\partial F}{\partial Q}\frac{\partial}{\partial y}\delta z = \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial Q}\delta z\right\} - \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial Q}\right\}\delta z$$

$$\begin{aligned}
\delta v = \iint_D &\left[\frac{\partial F}{\partial Z}\delta z - \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial P}\right\}\delta z - \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial Q}\right\}\delta z\right]_{\alpha=0} dxdy \\
&+ \iint_D \left[\frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial P}\delta z\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial Q}\delta z\right\}\right]_{\alpha=0} dxdy
\end{aligned}$$

$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ とすると，

$$\begin{aligned}
\delta v &= \iint_D \left[\frac{\partial F}{\partial Z} - \frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial P}\right\} - \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial Q}\right\}\right]_{\alpha=0} \delta z dxdy + \iint_D \left[\frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial P}\delta z\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial Q}\delta z\right\}\right]_{\alpha=0} dxdy \\
&= \iint_D \left(F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)\delta z dxdy + \iint_D \left[\frac{\partial}{\partial x}\left\{\frac{\partial F}{\partial p}\delta z\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial F}{\partial q}\delta z\right\}\right] dxdy
\end{aligned}$$

グリーンの公式により，

$$\begin{aligned}\delta v &= \iint_D \left(F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)\delta z dxdy + \int_C \left\{\frac{\partial F}{\partial p}\delta z\right\} dy - \int_C \left\{\frac{\partial F}{\partial q}\delta z\right\} dx \\ &= \iint_D \left(F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)\delta z dxdy\end{aligned}$$

$$F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q \equiv 0$$

$$\boxed{F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q = 0}$$

この偏微分方程式は**オストグラツキーの方程式**(*Ostrogradski equation*) または**オイラー・ラグランジェの方程式**(*Euler-Lagrange equation*) と呼ばれる．

グリーンの定理

$$\boxed{\iint_D \frac{\partial f}{\partial y}dxdy = -\int_C f dx, \iint_D \frac{\partial f}{\partial x}dxdy = \int_C f dx}$$ （証明略）

例題 1.

$$v(z(x,y)) = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] dxdy$$

のオストグラツキーの方程式は,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

例題 2.

$$v(z(x,y)) = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 2zf(x,y)\right] dxdy$$

のオストグラツキーの方程式は,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x,y)$$

例題 3.

$$S(z(x,y)) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$$

のオストグラツキーの方程式は,

$$\frac{\partial}{\partial x}\left(\frac{p}{\sqrt{1+p^2+q^2}}\right) + \frac{\partial}{\partial y}\left(\frac{q}{\sqrt{1+p^2+q^2}}\right) = 0$$

$\frac{p}{\sqrt{1+p^2+q^2}} = u, \frac{q}{\sqrt{1+p^2+q^2}} = v$ とすると,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial x}+\frac{\partial v}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial v}{\partial q}\frac{\partial q}{\partial y}=0$$

$$\frac{\sqrt{1+p^2+q^2}-\dfrac{p^2}{\sqrt{1+p^2+q^2}}}{1+p^2+q^2}\frac{\partial p}{\partial x}-2\cdot\frac{\dfrac{pq}{\sqrt{1+p^2+q^2}}}{1+p^2+q^2}\frac{\partial q}{\partial x}+\frac{\sqrt{1+p^2+q^2}-\dfrac{q^2}{\sqrt{1+p^2+q^2}}}{1+p^2+q^2}\frac{\partial q}{\partial y}=0$$

$$\left(\sqrt{1+p^2+q^2}-\frac{p^2}{\sqrt{1+p^2+q^2}}\right)\frac{\partial p}{\partial x}-\frac{2pq}{\sqrt{1+p^2+q^2}}\frac{\partial q}{\partial x}+\left(\sqrt{1+p^2+q^2}-\frac{q^2}{\sqrt{1+p^2+q^2}}\right)\frac{\partial q}{\partial y}=0$$

$$(1+q^2)\frac{\partial p}{\partial x}-2pq\frac{\partial q}{\partial x}+(1+p^2)\frac{\partial q}{\partial y}=0$$

$$\left\{1+\left(\frac{dz}{dy}\right)^2\right\}\frac{\partial^2 z}{\partial x^2}-2\cdot\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial x}\cdot\frac{\partial^2 z}{\partial xy}+\left\{1+\left(\frac{dz}{dx}\right)^2\right\}\frac{\partial^2 z}{\partial y^2}=0$$

（後略）

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